12 THREE-DIMENSIONAL GEOMETRY

IMPORTANT FORMULAE

- If direction cosines of a line are l, m, n then $l^2 + m^2 + n^2 = 1$.
- Direction cosines of the line passing through two points. $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\frac{x_2 - x_1}{PQ}$, $\frac{y_2 - y_1}{PQ}$, $\frac{z_2 - z_1}{PQ}$, where PQ

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- The numbers which are proportional to direction cosines of a line are called direction ratios of the line.
- If direction cosines of a line are l, m, n and direction ratios are a, b, c, then

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}; m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}; n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

- If l_1, m_1, n_1 and $l_2, m_2 n_2$ are the direction cosines of two lines and θ is the acute angle between them, then $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$
- If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines, then

$$\cos\theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

- Equation of a line through a given point $\vec{a}$ and parallel to $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$
- Let the point be (x_1, y_1, z_1) and l, m, n be the direction ratios then equation of the line :

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

- Vector equation of the line passing through two points whose position vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

- Cartesian equations of the line passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) are.

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

- If θ is the acute angle between two lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda \vec{b_2}$, then

$$\cos\theta = \left| \frac{\vec{b_1} . \vec{b_2}}{|\vec{b_1}| \, |\vec{b_2}|} \right|$$

- The shortest distance between two skew lines is a line segment which is perpendicular on both lines.
- Shortest distance, between two lines $\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu \vec{b_2}$ is $\left| \frac{(\vec{b_1} \times \vec{b_1}).(\vec{a_2} - \vec{a_1})}{|\vec{b_2} \times \vec{b_2}|} \right|$
- Shortest distance between two lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ is

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2}}$$

- Distance between two parallel lines $\vec{r} = \vec{a_1} + \lambda \vec{b}$ and $\vec{r} = \vec{a_2} + \mu \vec{b}$ is $\left| \frac{\vec{b} \times (a_2 - a_1)}{|\vec{b}|} \right|$.
- The equation and a plane whose distance from origin is d and normal unit vector from origin to the plane is n, in the vector form is $\vec{r}.\hat{n} = d$.
- If l, m, n are the direction cosines of the normal to the plane which is at distance d from the origin, then equation of the plane is $lx + my + nz = d$.
- Let the plane passes through a point A $(\vec{a})$. It is perpendicular to the vector $\vec{N}$. Then equation of the plane is $(\vec{r} - \vec{a}).\vec{N} = 0$.
- If a plane passes through (x_1, y_1, z_1) and perpendicular to the line with direction ratios free a, b, c, then the equation of the plane is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.
- Equation of the plane passing through three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

- Though three points whose position vectors are $\vec{a}, \vec{b}$ and $\vec{c}$ vector equation of the plane is

$$(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$$

- Equation of a plane which intersect the axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ life $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

- Let the equations of two planes be $\vec{r}.\vec{n_1} = d_1$ and $\vec{r}.\vec{n_2} = d_2$. The equation of the plane passing through the line of intersection of the plane is

$$\vec{r}(\vec{n_2}+\lambda\vec{n_2}) = d_1 + \lambda d_2.$$

- Equation of the plane passing through the line & intersection of planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is

$$(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$$

- The lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ are coplanar, if $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_1 & c_2 \end{vmatrix} = 0$

- The angles between the two planes is the angle between their normals. The angle q between the planes $\vec{r}.\vec{n_1} = d_1$ and $\vec{r}.\vec{n_2} = d_2$ is given by

$$\theta = \cos^{-1}\frac{|\vec{n_1}.\vec{n_2}|}{|\vec{n_2}|.|\vec{n_1}|}.$$

- Let the line and plane be $\vec{r} = \vec{a}+\lambda\vec{b}$ and $\vec{r}.\hat{n} = d$ respectively and θ be the angle between the plane and the line, then

$$\sin\theta = \frac{\vec{b}.\vec{n}}{|\vec{b}||\vec{n}|}$$

- The angle θ between two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is given by

$$\theta = \cos^{-1}\left|\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}.\sqrt{A_2^2+B_2^2+C_2^2}}\right|$$

- In vector form the distance of a point whose position vector is $\vec{a}$,, from the plane $\vec{r}.\hat{n} = d$ is $(d - \vec{a}.\hat{n})$.
- The distance of a point (x_1, y_1, z_1) from the plane $Ax + By + Cz + D =$ is $\left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2+B^2+C^2}}\right|$.

Multiple Choice Questions

1. The direction cosines of z-axis are : (*BSEB*, 2013)
   (a) 0, 0, 0 (b) 1, 0, 0
   (c) 0, 1, 0 (d) 0, 0, 1
2. Equation of a plane not cuts the co-ordinate axis at $(a, 0, 0)$ $(0, b, 0)$ and $(0, 0, c)$ is : (*BSEB*, 2013)
   (a) $ax + by + zr + d = 0$ (b) $ax + by + zx = 0$
   (c) $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 0$ (d) $\frac{x}{a}-\frac{y}{b}-\frac{z}{c} = 0$
3. The distance between (4, 3, 7) and (1, – 1, – 5) is : (*BSEB*, 2013)
   (a) 5 (b) 5 (c) 13 (d) none of these
4. If O be the origin and the co-ordinates of p be (1, 2, – 3), then the equation of the plane passing through p and perpendicular to OP is : (*BSEB*, 2013)
   (a) $x + 2y - 3z - 14 = 0$ (b) $x + 2y + 3z = 14$
   (c) $x - 2y + 3z = 14$ (d) $x - 2y - 3z = 14$
5. The direction cosines of the normal to the plane $2x - 3y - 6z - 3 = 0$ are : (*BSEB*, 2010)
   (a) $\frac{2}{7}, -\frac{3}{7}, -\frac{6}{7}$ (b) $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$
   (c) $-\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$ (d) none of these
6. The line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$ (*BSEB*, 2010)
   (a) passes through (5, – 4, 6)
   (b) has direction cosines 3, 7, 2
   (c) is perpendicular to $3x + 7y - 2z = 0$
   (d) none of these
7. For a straight line having direction coming l, m, n, $l^2 + m^2 + n^2$: (*BSEB*, 2014)
   (a) 0 (b) 1 (c) – 1 (d) 2
8. The necessary condition for the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ to be parallel to the plane $ax + by + cx + d = 0$ is : (*BSEB*, 2014)
   (a) $al + bn + cn = 0$ (b) $al + bn + cm = 1$
   (c) $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$ (d) none of these
9. If a straight line makes equal angles with the co-ordinate axes, then its direction ratios are :
   (a) 1, 2, 3 (b) 3, 1, 2
   (c) 3, 2, 1 (d) 1, 1, 1
10. The co-ordinates of the point, which is equidistant from the points (0, 0, 0), $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ are :
    (a) $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$ (b) $\left(-\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$
    (c) $\left(\frac{a}{2}, -\frac{b}{2}, \frac{c}{2}\right)$ (d) $\left(\frac{a}{2}, \frac{b}{2}, -\frac{c}{2}\right)$
11. The direction cosines of the y-axis are : (*BSEB*, 2015)
    (a) (0, 0, 0) (b) (1, 0, 0) (c) (0, 1, 0) (d) (0, 0, 1)
12. If the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ is parallel to the plane $ax + by + cz + d = 0$, then : (*BSEB*, 2015)
    (a) $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$ (b) $al + bm + cn = 0$
    (c) $al^2 + bm^2 + cn^2 = 0$ (d) $a^2l^2 + b^2m^2 + c^2n^2 = 0$
13. If the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular to each other, then : (*BSEB*, 2015)
    (a) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$ (b) $\frac{a_1}{a_2} + \frac{b_1}{b_2} + \frac{c_1}{c_2} = 0$
    (c) $a_1a_2 + b_1b_2 + c_1c_2 = 0$ (d) $a_1^2a_2^2 + b_1^2b_2^2 + c_1^2c_2^2 = 0$

14. The distance of the plane $2x + 3y + 6z + 7 = 0$ from the point $(2, -3, -1)$ is : (*BSEB*, 2015)

(a) 4 (b) 3 (c) 2 (d) $\frac{1}{5}$

15. The directions ratios of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are : (*BSEB*, 2015)

(a) $x_1 + x_2, y_1 + y_2, z_1 + z_2$

(b) $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

(c) $\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}$

(d) $x_2 - x_1, y_2 - y_1, z_2 - z_1$

16. The co-ordinates of the mid-point of the line segment joining the points (2, 3, 4) and (8, – 3, 8) are : (*BSEB*, 2015)

(a) (10, 0, 12)(b) (5, 6, 0) (c) (6, 5, 0) (d) (5, 0, 6)

17. If the direction cosines of two straight lines are l_1, m_1, n_1 and l_2, m_2, n_2 then the cosine of the angle θ between them or $\cos\theta$ is : (*BSEB*, 2015)

(a) $(l_1 + m_1 + n_1)(l_2 + m_2 + n_2)$

(b) $\frac{l_1}{l_2} + \frac{m_1}{m_2} + \frac{n_1}{n_2}$

(c) $l_1 l_1 + m_1 m_2 + n_1 n_1$

(d) $\frac{l_1 + m_1 + n_1}{l_2 + m_2 + n_2}$

18. The direction ratios of the normal to the plane $7x + 4y - 2z + 5 = 0$ are : (*BSEB*, 2015)

(a) 7, 4, 5 (b) 7, 4, – 2 (c) 7, 4, 2 (d) 0, 0, 0

Ans. 1. (d), 2. (c), 3. (c), 4. (a), 5. (a), 6. (a), 7. (b), 8. (a), 9. (d), 10. (a), 11. (c), 12. (a), 13. (c), 14. (a), 15. (d), 16. (d), 17. (c), 18. (b).

Very Short Answer Type Questions

Q. 1. If a line makes angles 90°, 60° and 30° with the positive direction of *x*, *y* and *z*-axis respectively, find its direction cosines.

(*Raj*, 2014; *Uttrakhand*, 2014)

Solution

$l = \cos 90° = 0, m = \cos 60° = \frac{1}{2}, n = \cos 30° = \frac{\sqrt{3}}{2}$

$\therefore$ Direction cosines of the line are $0, \frac{1}{2}, \frac{\sqrt{3}}{2}$.

Q. 2. Write the cartesian of the straight line through the point (α, β, γ) and parallel to *z*-axis.

[*AI CBSE*, 2014 (*Comptt.*)]

Solution

$\because$ Line is parallel to *z*-axis and d.c.'s of *z*-axis are 0, 0, 1

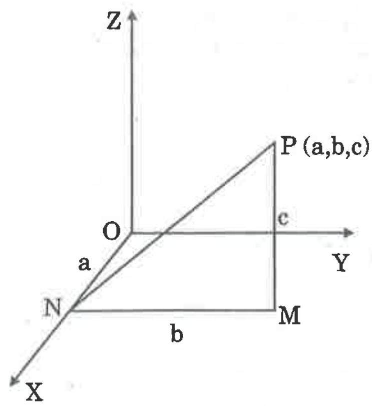
$\therefore$ D.C.'s of the line are 0, 0, 1

Also the line passes through the point (α, β, γ)

$\therefore$ Equations of the line are

$$\frac{x-\alpha}{0} = \frac{y-\beta}{0} = \frac{z-\gamma}{1}$$

Q. 3. Write the distance of a point *p* (*a*, *b*, *c*) from *x*-axis. [*CBSE*, 2014 (*Comptt.*)]



Solution

Let *m* be the foot of the perpendicular from p or XOY plane.

Let N be the foot OX the perpendicular from M on *x*-axis.

Then, ON = *a*, NM = *b*, MP = *c*

$\therefore$ Distance of *p* from *x*-axis = NP

$$= \sqrt{NM^2 + MP^2}$$

$$= \sqrt{b^2 + c^2}$$

Q. 4. If the cartesian equations of a line are : $\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$, write the vector equation for the line. (*AI CBSE*, 2014)

Solution

$$\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$$

$$\Rightarrow \frac{x-3}{-5} = \frac{y-(-4)}{7} = \frac{z-3}{2}$$

$$\therefore \vec{a} = 3\hat{i} - \hat{j} + 3\hat{k}$$

$$\vec{b} = -5\hat{i} + 7\hat{j} + 3\hat{k}$$

$\therefore$ vector equation of the line is

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

$$\Rightarrow \vec{r} = (3\hat{i} - 4\hat{j} + 3\hat{k}) + \lambda(-3\hat{i} + 7\hat{j} + 2\hat{k})$$

where λ is a parameter.

Q. 5. Write the vector equation of a line passing through the point (1, – 1, 2) and parallel to the line whose equations are

$$\frac{x-3}{1} = \frac{y-1}{2} = \frac{z+1}{-2}$$ [*CBSE*, 2013 (*Comptt.*)]

Solution

Here, $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$

$$\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$$

Hence, the vector equation of the line is

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

$$\Rightarrow \vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + (\hat{i} + 2\hat{j} - 2\hat{k})$$

where λ is a parameter.

Q. 6. Find the cartesian equations of the line which passes through the point (– 2, 4, – 5) and is parallel to the line

$$\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$$ (*CBSE*, 2013)

Solution

$$\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$$

$$\Rightarrow \frac{x+3}{3} = \frac{y-4}{-5} = \frac{z+8}{6}$$

cartesian equation of the line are

$$\frac{x+2}{3} = \frac{y-4}{-5} = \frac{z+5}{6}$$

Q. 7. Find the equations of the straight line which passes through the point (0, – 1, 4) and is parallel to the straight line.

$$\frac{-x-2}{1} = \frac{y+3}{7} = \frac{2z-6}{3}$$

$$\Rightarrow \frac{x+2}{-1} = \frac{y+3}{7} = \frac{z-3}{3/2}$$

$$\Rightarrow \frac{x+2}{-2} = \frac{y+3}{14} = \frac{z-3}{3}$$

Hence the equations of the required straight line are

$$\frac{x-0}{-2} = \frac{y+1}{14} = \frac{z-4}{3}$$

$$\Rightarrow \frac{x}{-2} = \frac{y+1}{14} = \frac{z-4}{3}$$

Q. 8. If a line has direction ratios 2, – 1, – 2, determine its direction cosines. (*USEB*, 2013)

Solution

$$\sqrt{(2)^2+(-1)^2+(-2)^2} = \sqrt{9} = 3$$

∴ Direction cosines are

$$\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}$$

Q. 9. Find the direction cosines of the unit vector perpendicular to the plane $\vec{r}.(6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$. (*Raj Board,*, 2013)

Solution

A vector perpendicular the plane is $(6\hat{i} - 3\hat{j} - 2\hat{k})$

∴ A unit vector perpendicular to the plane is

$$\frac{6\hat{i}-3\hat{j}-2\hat{k}}{\sqrt{(6)^2+(-3)^2+(-2)^2}}$$

$$= \frac{6\hat{i}-3\hat{j}-2\hat{k}}{7}$$

$$= \frac{6}{7}\hat{i} - \frac{3}{7}\hat{j} - \frac{2}{7}\hat{k}$$

∴ Direction cosines of the unit vector are $\frac{6}{7}, -\frac{3}{7}, -\frac{2}{7}$.

Q. 10. Find the value of α, if the straight line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-4}{\lambda}$ and $\frac{x-2}{1} = \frac{y-5}{3} = \frac{z-1}{-7}$ are perpendicular to each other. (*JAC*, 2014)

Solution

If the given straight lines are perpendicular to each other, then

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\Rightarrow (2)(1) + (4)(3) + (\lambda)(-7) = 0$$

$$\Rightarrow 2 + 12 - 7\lambda = 0$$

$$\Rightarrow 14 - 7\lambda = 0$$

$$\Rightarrow 7\lambda = 14$$

$$\Rightarrow \lambda = 2$$

Q. 11. The lines $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are perpendicular to each other. Find the value of *p*. (*Raj. Board,*, 2013)

Solution

The given lines are

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$$

$$\Rightarrow \frac{x-1}{-3} = \frac{y-2}{2/7\,p} = \frac{z-3}{2} \quad ...(1)$$

and $$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

$$\Rightarrow \frac{x-1}{-3/7p} = \frac{y-5}{1} = \frac{z-6}{-5} \quad ...(2)$$

If the lines (1) and (2) are perpendicular to each other, then

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\Rightarrow (-3)\left(-\frac{3}{7}p\right) + \left(\frac{2}{7}p\right)(1) + (2)(-5) = 0$$

$$\Rightarrow \frac{9p}{7} + \frac{2p}{7} - 10 = 0$$

$$\Rightarrow \frac{11p}{7} - 10 = 0$$

$$\Rightarrow \frac{11p}{7} = 10$$

$$\Rightarrow p = \frac{70}{11}$$

Q. 12. Find the acute angle between the planes 2*x* – *y* + *z* + 8 = 0 and *x* + *y* + 2*z* – 14 = 0. (*BSEB*, 2014)

Solution

$$\cos\theta = \frac{(2)(1)+(-1)(1)+(1)(2)}{\sqrt{(2)^2+(-1)^2+(1)^2}\sqrt{(1)^2+(1)^2+(2)^2}}$$

$$= \frac{3}{6}$$

$$= \frac{3}{6} = \cos 60°$$

$\Rightarrow \quad \theta = 60°$

Hence the required acute angle is 60°

Q. 13. Find the angle between the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$. *(Raj. Board, 2014)*

Solution

$$\cos\theta = \frac{(2)(3)+(1)(-6)+(-2)(-2)}{\sqrt{(2)^2+(1)^2+(-2)^2}\sqrt{(3)^2+(-6)^2+(-2)^2}}$$

$$= \frac{4}{3\times 7} = \frac{4}{21}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{4}{21}\right)$$

Hence the required angle is $\cos^{-1}\left(\frac{4}{21}\right)$.

Q. 14. Find the angle between the lines $\frac{5-x}{3} = \frac{y+3}{-4}$, $z = 7$ and $\frac{x}{1} = \frac{1-y}{2} = \frac{z-6}{2}$. *(JAC, 2013)*

Solution

The given lines are

$$\frac{5-x}{3} = \frac{y+3}{-4} = \frac{z-7}{0}$$

$$\Rightarrow \quad \frac{x-9}{-3} = \frac{y+3}{-4} = \frac{z-7}{0}$$

and

$$\frac{x}{1} = \frac{1-y}{2} = \frac{z-6}{2}$$

$$\Rightarrow \quad \frac{x}{1} = \frac{y-1}{-2} = \frac{z-6}{2} \quad ...(2)$$

$$\cos\theta = \frac{(-3)(1)+(-4)(-2)+(0)(2)}{\sqrt{(-3)^2+(-4)^2+(0)^2}\sqrt{(1)^2+(-2)^2+(2)^2}}$$

$$= \frac{5}{\sqrt{25}\sqrt{9}} = \frac{5}{5\times 3} = \frac{1}{3}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

Hence the required angle is $\cos^{-1}\left(\frac{1}{3}\right)$.

Q. 15. Find the acute angle between two lines that have the direction ratios (1, 1, 0) and (2, 1, 2).

Solution

$$\cos\theta = \frac{(1)(2)+(1)(1)+(0)(2)}{\sqrt{1^2+1^2+0^2}\sqrt{2^2+1^2+2^2}}$$

$$= \frac{2+1+0}{\sqrt{2}\times 3} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \theta = 45°$$

Hence the required acute angle is 45°.

Q. 16. Find the length of the perpendicular drawn from the origin to the plane $2x - 3y + 6z + 21 = 0$. *(AI CBSE, 2013)*

Solution

Length of the perpendicular

$$= \frac{2(0)-3(0)+6(0)+21}{\sqrt{(2)^2+(-3)^2+(6)^2}} = \frac{21}{7}$$

= 3 units

Short Answer Type Questions

Q. 1. Find the co-ordinates of the point where the line through the points A (3, 4, 1) and B (5, 1, 6) arises the xy-plane. *(USEB, 2014)*

Solution

Equation of the line through the points A (3, 4, 1) and B (5, 1, 6) are :

$$\frac{x-3}{5-3} = \frac{y-4}{1-4} = \frac{z-1}{6-1}$$

$$\Rightarrow \quad \frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5}$$

Equation of xy-plane is $z = 0$.

If cosines the xy-plane at the point of which $z = 0$, therefore, putting $z = 0$ in the equation of the lines, then

$$\frac{x-3}{2} = \frac{y-4}{-3} = \frac{0-1}{5} = -\frac{1}{5}$$

$$\Rightarrow \quad x = 3 - \frac{2}{5} = \frac{13}{5}$$

$$y = 4 + \frac{3}{5} = \frac{23}{5}$$

Hence, the required point is $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

Q. 2. Show that the lines $\frac{5-x}{-4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-5}{3}$ are coplanar. *(CBSE, 2014)*

Solution

The co-ordinate of coplanar is

$$\begin{vmatrix} 8-9 & 4-7 & 5-(-3) \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$$

The given lines are $\frac{5-x}{-4} = \frac{y-7}{4} = \frac{z+3}{-5}$

and $\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-5}{3}$

$$\Rightarrow \quad \frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$$

and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$

$$\Rightarrow \quad \begin{vmatrix} 3 & -3 & 8 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$$

$\Rightarrow 3(12+5) - 3(-35-12) + 8(4-28) = 0$

$\Rightarrow \quad 51 + 161 - 176 = 0$

$\Rightarrow \quad 0 = 0$, which is true

Hence the given lines are coplanar.

Q. 3. Find the equation of the plane that contains the point (1, – 1, 2) and is perpendicular to each of the planes $2x + 3y - 2z = 5$ and $x + 2x - 3z = 8$. (*USEB*, 2014)

Solution

Any plane through the point (1, – 1, 2) is given by

$A(x-1) + B(y+1) + C(z-2) = 0$ ...(1)

whose A, B, C are the d.v.'s of the normal to the plane

∵ (1) is perpendicular to the plane

$$2x + 3y - 2z = 5$$

$\therefore \quad 2A + 3B - 2C = 0$ ...(2)

∵ (1) is perpendicular to the plane

$$x + 2y - 3z = 8$$

$\therefore \quad A + 2B - 3C = 0$ ...(3)

Eliminating A, B, C from (1), (2) and (3) determinantically, we get

$$\begin{vmatrix} x-1 & y-1 & z-2 \\ 2 & 3 & -2 \\ 1 & 2 & -3 \end{vmatrix} = 0$$

$\Rightarrow (-9+4)(x-1) + (-2+6)(y-1) + (4-3)(z-2) = 0$

$\Rightarrow \quad -9x + 5 + 4y - 4 + z - 2 = 0$

$\Rightarrow \quad -9x + 4y + z = 1$

which is the required equation of the plane.

Q. 4. Show that the line $\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-9}{5}$ and $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$ are coplanar.

[(*USEB*, 2013; *CBSE*, 13 (*Comptt.*)]

Solution

The co-ordinates of coplanar

$$\begin{vmatrix} (-1)-(-3) & 2-1 & 5-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0$$

$\Rightarrow \quad 2(5-10) + 1(-5+15) + 0(-6+1) = 0$

$\Rightarrow \quad -10 + 10 + 0 = 0$

$\Rightarrow \quad 0 = 0$ which is true.

Hence the given lines are coplanar.

Q. 5. Write the vector equation of the plane passing through the point (*a*, *b*, *c*) and parallel to the plane $\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 2$. (*CBSE*, 2014)

Solution

Here, $\vec{d} = a\hat{i} + b\hat{j} + c\hat{k}$

$$\vec{n} = \hat{i} + \hat{j} + \hat{k}$$

∴ vector equation of the plane is

$$(\vec{r} - \vec{d}).\vec{n} = 0$$

$$\Rightarrow \quad \vec{r}\,\vec{h} - \vec{d}.\vec{n} = 0$$

$$\Rightarrow \quad \vec{r}\,\vec{n} = \vec{d}.\vec{n}$$

$$\Rightarrow \quad \vec{r}.(\hat{i} + \hat{j} + \hat{k}) = (a\hat{i} + b\hat{j} + c\hat{k}).(\hat{i} + \hat{j} + \hat{k})$$

$$\Rightarrow \quad \vec{r}.(\hat{i} + \hat{j} + \hat{k}) = a + b + c$$

Q. 6. Find the vector and cartesian equation of the line passing through the point (2, 1, 3) and perpendicular to the lines

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$$

and $\frac{x}{-3} = \frac{y}{2} = \frac{z}{5}$ (*AI CBSE*, 2014)

Solution

Let *l*, *m*, *n* be the d.c.'s of the line.

∵ line is perpendicular to the lines

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$$

and $\frac{x}{-3} = \frac{y}{2} = \frac{z}{5}$

$\therefore \quad l(1) + m(2) + n(3) = 0$

$\quad l(-3) + m(2) + n(5) = 0$

$\therefore \quad \frac{l}{10-6} = \frac{m}{-9-5} = \frac{n}{2+6}$

$\Rightarrow \quad \frac{l}{4} = \frac{m}{-14} = \frac{n}{8}$

$\Rightarrow \quad \frac{l}{2} = \frac{m}{-1} = \frac{n}{4} = \lambda$ (say)

$\Rightarrow \quad l = 2\lambda, m = -7\lambda, n = 4\lambda$

∴ Equation of the line are

$$\frac{y-2}{2\lambda} = \frac{y-1}{-7\lambda} = \frac{z-3}{4\lambda}$$

$$\Rightarrow \quad \frac{x-2}{2} = \frac{y-1}{-7} = \frac{z-3}{4}$$

Q. 7. Find the volume of *p*, so that the lines l_1 : $\frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$ and l_2 : $\frac{7-7y}{3p} = \frac{y-5}{1} = \frac{b-z}{5}$ are perpendicular to each other. Also find the equations of a line passing through a point (3, 2, – 4) and parallel to line l_1. (*AI CBSE*, 2014)

Solution

The given lines are

$$l_1 : \frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$$

$$\Rightarrow \quad l_1 : \frac{1-x}{-3} = \frac{y-2}{p/7} = \frac{z-3}{2} \quad ...(1)$$

$$\text{and} \quad l_2 : \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

$$\Rightarrow \quad l_2 : \frac{1-x}{-3/7p} = \frac{y-5}{1} = \frac{z-6}{-5} \quad ...(2)$$

If the lines l_1 and l_2 ane perpendicular to each other, then

$$(-3)\left(-\frac{3}{7}p\right) + \left(\frac{p}{7}\right)(-1) + (2)(-5) = 0$$

$$\Rightarrow \quad \frac{9}{7}p + \frac{p}{7} - 10 = 0$$

$$\Rightarrow \quad \frac{10p}{7} - 10 = 0$$

$$\Rightarrow \quad 10p = 70$$

$$\Rightarrow \quad p = 7$$

$\therefore$ DR's of line l_1 are – 3, 1, 2.

Hence, the equation of a line passing through a point (3, 2, – 4) and parallel to the line h are $\frac{x-3}{-3} = \frac{y-2}{1} = \frac{z-2}{2}$

Q. 8. Find the vector equation of the plane through the points (2, 1, – 1) and (– 1, 3, 4) and perpendicular to the plane $x - 2y + 4z = 10$. (*AI CBSE*, 2013)

Solution

Any plane through the point (2, 1, – 1) is given by

$A(x-2) + B(y-1) + c(z+1) = 0$ ...(1)

whose A, B, C are the d.r.'s of the round to the plane.

If plane (1) passes through the point (– 1, 3, 4), then

$$A(-1-2) + B(3-1) + C(4+1) = 0$$

$$\Rightarrow \quad A(-3) + B(2) + C(5) = 0$$

If plane (i) is perpendicular to the plane $x - 2y + 4z = 10$, then

$A(1) + B(-2) + C(4) = 0$ ...(3)

Eliminating A, B, C from (1), (2) and (3) determinantically, we get

$$\begin{vmatrix} x-2 & y-1 & z+1 \\ -3 & 2 & 5 \\ 1 & -2 & 4 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-2)(8+10) + (y-1)(5+12) + (z+1)(6-2) = 0$$

$$\Rightarrow \quad 18(x-2) + 17(y-1) + 4(z+1) = 0$$

$$\Rightarrow \quad 18x + 17y + 4z = 49$$

Q. 9. The position vectors of the two points A and B are $3\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} - 2\hat{j} - 4\hat{k}$ respectively. Find the vector equation of the plane passing through B and perpendicular to $\vec{AB}$. (*JAC*, 2014)

Solution

Here, $\vec{d} = \hat{i} - 2\hat{j} - 4\hat{k}$

$$\vec{n} = (\hat{i} - 2\hat{j} - 4\hat{k}) - (3\hat{i} + \hat{j} + 2\hat{k})$$

$$= -2\hat{i} - 3\hat{j} - 6\hat{k}$$

Hence, the vector equation of the required plane is

$$\Rightarrow (\vec{r} - \vec{d}).\vec{n} = 0$$

$$\Rightarrow \quad \vec{r}.\vec{n} = \vec{d}.\vec{n}$$

$$\Rightarrow \vec{r}.(-2\hat{i} - 3\hat{j} - 6\hat{k}) = (\hat{i} - 2\hat{j} - 4\hat{k}).(-2\hat{i} - 3\hat{j} - 6\hat{k})$$

$$= -2 + 6 + 24 = 28$$

$$\Rightarrow \vec{r}.(2\hat{i} + 3\hat{j} + 6\hat{k}) + 28 = 0$$

Q. 10. Find the equation of the plane passing therefore the line of intersection of the planes $\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r}.(2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ are parallel to x-axis. [*AI CBSE*, 2014 (*Comptt.*)]

Solution

Any plane passing through the line of intersection of the planes

$\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 1$ ...(1)

and $\vec{r}.(2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ ...(2)

is given by

$$\vec{r}.(\hat{i} + \hat{j} + \hat{k}) - 1 + \lambda[\vec{r}.(2\hat{i} + 3\hat{j} - \hat{k}) + 4] = 0$$

where λ is a parameter.

$\Rightarrow \vec{r}\{(1+2\lambda)\hat{i} + (1+3\lambda)\hat{j} + (1-\lambda)\hat{k}\} - 1 + 4\lambda = 0$ ...(3)

Here, $\vec{n} = (1+2\lambda)\hat{i} + (1+3\lambda)\hat{j} + (1-\lambda)\hat{k}$

$\therefore$ DR's of the normal are $1 + 2\lambda$, $1 + 3\lambda$, $1 - \lambda$

$\because$ plane (3) is parallel to x-axis.

$\therefore$ Normal to the plane (3) is perpendicular to x-axis.

$\therefore \quad (1+2\lambda)(1) + (1+3\lambda)(0) + (1-\lambda) = 0$

$$\Rightarrow \quad 1 + 2\lambda = 0$$

$$\Rightarrow \quad \lambda = -\frac{1}{2}$$

Putting the value of λ is (3), we get

$$\Rightarrow \quad \vec{r}\left\{-\frac{1}{2}\hat{j} + \frac{3}{2}\hat{k}\right\} - 3 = 0$$

$$\Rightarrow \quad \vec{r}(-\hat{j} + 3\hat{k}) = 6$$

$$\Rightarrow \quad \vec{r}(\hat{j} - 3\hat{k}) + 6 = 0$$

$$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}).(\hat{j} - 3\hat{k}) + 6 = 0$$

$$\Rightarrow \quad y - 3z + 6 = 0$$

Q. 11. Find the vector equation of the line passing through the point (1, 2, 3) and parallel to the planes $\vec{r}.(\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\vec{r}.(3\hat{i} + \hat{j} + \hat{k}) = 6$. (*AI CBSE*, 2013)

Solution

Let the line be parallel to the vector

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

Since the line is parallel to the planes $\vec{r}.(\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\vec{r}.(3\hat{i} + \hat{j} + \hat{k}) = 6$, therefore the line will be perpendicular to the normal to each of these planes, therefore,

$$b_1 - b_2 + 2b_3 = 0$$

$$3b_1 + b_2 + b_3 = 0$$

$$\therefore \quad \frac{b_1}{-1-2} = \frac{b_2}{6-1} = \frac{b_3}{1+3}$$

$$\Rightarrow \quad \frac{b_1}{-3} = \frac{b_2}{5} = \frac{b_3}{4}$$

$\therefore \qquad \vec{b} = -3\hat{i} + 5\hat{j} + 4\hat{k}$

Hence, the vector equation of the required line is

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

$$\Rightarrow \quad \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-3\hat{i} + 5\hat{j} + 4\hat{k})$$

where λ is a parameter.

Q. 12. Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Also, find their point of intersection. (*CBSE*, 2014)

Solution

The given lines are

$$\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} = r \text{ (say)} \qquad ...(1)$$

and $$\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5} = R \text{ (say)} \qquad ...(2)$$

Any point on line (1) is $(-1 + 3r, -3 + 5r, -5 + 7r)$

Any point on line (2) is $(2 + R, 4 + 3R, 6 + 5R)$

If lines (1) and (2) intersect, then at the point of intersection for some value of r and R, we have

$$-1 + 3r = 2 + R$$
$$-3 + 5r = 4 + 3R$$
$$-5 + 7r = 6 + 5R$$

$$\Rightarrow \quad 3r - R = 3 \qquad ...(3)$$
$$5r - 3R = 7 \qquad ...(4)$$
$$7r - 5R = 11 \qquad ...(5)$$

Solving equations (3) and (4) for r and R, we get

$$r = \frac{1}{2}, R = -\frac{3}{2}$$

These values of r and R satisfy equation (5) as

$7\left(\frac{1}{2}\right) - 5\left(-\frac{3}{2}\right) = 11$ is true

Hence the given lines (1) and (2) intersect.

Point of intersection : Put $r = \frac{1}{2}$, we have

$$-1 + 3r = -1 + \frac{3}{2} = \frac{1}{2}$$
$$-3 + 5r = -3 + \frac{5}{2} = -\frac{1}{2}$$
$$-5 + 7r = -5 + \frac{7}{2} = -\frac{3}{2}$$

Hence the point of intersection is $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$

Q. 13. Find the vector and cartesian forms of the equation of the plane passig through the point (1, 2, – 4) and parallel to the lines : $\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$ and $\vec{r} = \hat{i} - 3\hat{j} + 5\hat{k} + \mu(\hat{i} + \hat{j} - \hat{k})$. Also, find the distance of the point (9, – 8, – 10) from the plane thus obtained. [*CBSE*, 2014 (*Comptt.*)]

Solution

Here, $\quad \vec{d} = \hat{i} + 2\hat{j} - 4\hat{k}$

$$\vec{r} = \vec{n_1} \times \vec{n_2}$$
$$= (2\hat{i} + 3\hat{j} + 6\hat{k}) \times (\hat{i} + \hat{j} - \hat{k})$$
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix}$$
$$= (-3 - 6)\hat{i} + (6 + 2)\hat{j} + (2 - 3)\hat{k}$$
$$= -9\hat{i} + 8\hat{j} - \hat{k}$$

Hence, the equation of the required plane is

$$(\vec{r} - \vec{d}).\vec{n} = 0$$
$$\Rightarrow \quad \vec{r}.\vec{n} - \vec{d}.\vec{n} = 0$$
$$\Rightarrow \quad \vec{r}.\vec{h} = \vec{d}.\vec{h}$$
$$\Rightarrow \quad \vec{r}.(-9\hat{i} + 8\hat{j} - \hat{k}) = (\hat{i} + 2\hat{j} - 4\hat{k}).(-9\hat{i} + 8\hat{j} - \hat{k})$$
$$= (1) \times (-9) + (2)(8) + (-4)(-1)$$
$$= -9 + 16 + 4$$
$$= 11$$

$$\Rightarrow \vec{r}.(9\hat{i} - 8\hat{j} + \hat{k}) + 11 = 0 \qquad ...(1)$$

which gives the vector form of the equation of the plane.

$$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}).(9\hat{i} - 8\hat{j} + \hat{k}) + 11 = 0$$
$$\Rightarrow \quad 9x - 8y + z + 11 = 0 \qquad ...(2)$$

which lines the cartesian form of the equation of the plane.

Distance of the point (9, – 8, – 10) from the plane (2)

$$= \frac{9(9) - 8(-8) + (10) + 11}{\sqrt{(9)^2 + (-8)^2 + (1)^2}}$$
$$= \frac{81 + 64 - 10 + 11}{\sqrt{81 + 64 + 1}} = \frac{146}{\sqrt{146}}$$
$$= \sqrt{146} \text{ units}$$

Q. 14. Find the distance between the point (7, 2, 4) and the plane determined by the points A (2, 5, – 3), B (– 2, – 3, 5) and C (5, 3, – 3). (*CBSE*, 2014)

Solution

Any plane passing through the point A (2, 5, – 3) is given by

$$a(x - 2) + b(y - 5) + c(z + 3) = 0 \qquad ...(1)$$

where a, b, c are the d.r.'s of the normal to the plane.

If (1) passes through B (– 2, – 3, 5) and C (5, 3, 3), we have

$$a(-2 - 2) + b(-3 - 5) + c(5 + 3) = 0$$
$$\Rightarrow \quad -4a - 8b + 8c = 0 \qquad ...(2)$$

and, $a(5 - 2) + b(3 - 5) + c(-3 + 3) = 0$

$$\Rightarrow \quad 3a - 2b + 0c = 0 \qquad ...(3)$$

eliminating a, b, c, eterminantically from equations (1), (2) and (3), we get

$$\begin{vmatrix} x-2 & y-5 & z+3 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

$\Rightarrow (0 + 16)(x - 2) + (24 - 0)(y - 5) + (8 + 24)(z + 3) = 0$

$\Rightarrow \quad 16(x - 2) + 24(y - 5) + 32(z + 3) = 0$

$\Rightarrow \quad 2(x - 2) + 3(y - 5) + 4(z + 3) = 0$

$\Rightarrow 2x + 3y + 4z - 7 = 0$ which is the equation of the plane ABC.

Distance of the point (7, 2, 4) from this plane

$$= \frac{2(7) + 3(2) + 4(4) - 7}{\sqrt{2^2 + 3^2 + 4^2}}$$

$$= \frac{14 + 6 + 16 - 7}{\sqrt{4 + 9 + 16}}$$

$$= \frac{29}{\sqrt{29}} = \sqrt{29} \text{ units}$$

Q. 15. A line passes through (2, – 1, 3) and is perpendicular to the lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda (2\hat{i} - 2\hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu (\hat{i} + 2\hat{j} + 2\hat{k})$, obtain its equations in vector and cartesian form.

(*AI CBSE*, 2014)

Solution

Here, $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$

$$\vec{b} = \vec{n}_1 \times \vec{n}_2$$

$$= (2\hat{i} - 2\hat{j} + \hat{k}) \times (\hat{i} + 2\hat{j} + 2\hat{k})$$

$$= \begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix}$$

$$= (-4 - 2)\hat{i} + (1 - 4)\hat{j} + (4 + 2)\hat{k}$$

$$= -6\hat{i} - 3\hat{j} + 6\hat{k}$$

Hence, the equation of the line in the vector form is

$\vec{r} = \vec{a} + t\vec{b}$ [where t is a parameter,]

$\Rightarrow \quad \vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(-6\hat{i} - 3\hat{j} - 6\hat{k})$

$\Rightarrow x\hat{i} + y\hat{j} + z\hat{k} = (2 - 6t)\hat{i} + (-1 - 3t)\hat{j} + (3 - 6t)\hat{k}$

Equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ on both sides, we get

$x = 2 - 6t, y = -1 - 3t, z = 3 - 6t$

$$\Rightarrow \quad \frac{x-2}{-6} = t$$

$$\frac{y+1}{-3} = t$$

$$\frac{z-3}{6} = t$$

$$\Rightarrow \quad \frac{x-2}{-6} = \frac{y+1}{-3} = \frac{z-3}{6}$$

which are the cartesian equation of the required line.

Q. 16. Find the value of $\hat{i}(\hat{j} \times \hat{k}) + \hat{j}(\hat{k} \times \hat{i}) + \hat{k}(\hat{i} \times \hat{j})$

(*JAC*, 2015)

Solution :

$\hat{i}(\hat{j} \times \hat{k}) + \hat{j}(\hat{k} \times \hat{i}) + \hat{k}(\hat{i} \times \hat{j})$

[$\because \hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$ and $\hat{j} \times \hat{i} = -\hat{i} \times \hat{j}$]

$= \hat{i}(\hat{i}) + \hat{j}.(\hat{j}) + \hat{k}.(\hat{k})$

$= 1 + 1 + 1 \quad [\because \hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1]$.

$= 3$

Q. 17. Find the area of a parallelogram whose adjacent sides are the vectors $\vec{a} = \hat{i} - 2\hat{i} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 4\hat{k}$.

(*JAC*, 2015)

Solution :

$\therefore$ Vector area of parallelogram $= \vec{a} \times \vec{b}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & -1 & 4 \end{vmatrix}$$

$$= \hat{i}(8 + 3) - \hat{j}(4 - 6) + \hat{k}(-1 + 4)$$

$$= 11\hat{i} + 2\hat{j} + 3\hat{k}$$

$\therefore$ Area of parallelogram

$$= |\vec{a} \times \vec{b}|$$

$$= |11\hat{i} + 2\hat{j} + 3\hat{k}|$$

$$= \sqrt{(11)^2 + (2)^2 + (3)^2}$$

$$= \sqrt{121 + 4 + 9}$$

$$= \sqrt{134} \text{ square units}$$

Q. 18. Find the angle between the vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$ (*USEB*, 2015)

Solution :

Let the angle between two vectors is θ, then

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\therefore \quad \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|}$$

$$= \frac{(\hat{i} + \hat{j} + \hat{k}).(\hat{i} + \hat{j} - \hat{k})}{|\hat{i} + \hat{j} + \hat{k}|.|\hat{i} + \hat{j} - \hat{k}|}$$

$$= \frac{(1 + 1 - 1)}{\left\{\sqrt{1^2 + 1^2 + 1^2}\right\}\left\{\sqrt{1^2 + 1^2 + (-1)^2}\right\}}$$

$$= \frac{1}{\sqrt{3}.\sqrt{3}}$$

$$= \frac{1}{3}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{3}\right)$$

Q. 19. A line has direction ratios 2, – 1, – 2 find the direction cosines. (*USEB*, 2015)

Solution :

Since direction ratios are 2, – 1, – 2

Thus, direction cosines will be

$$l = \frac{2}{\sqrt{2^2+(-1)^2(-2)^2}},\ m = \frac{-1}{\sqrt{(2)^2+(-1)^2+(-2)^2}},$$

$$\text{and } n = \frac{-2}{\sqrt{(2)^2+(-1)^2+(-2)^2}}$$

$$\Rightarrow \quad l = \frac{2}{\sqrt{4+4+1}},\ m = \frac{-1}{\sqrt{4+1+4}},\ n = \frac{-2}{\sqrt{4+1+4}}$$

$$\Rightarrow \quad l = \frac{2}{3},\ m = \frac{-1}{3},\ n = \frac{-2}{3}$$

Q. 20. Find the equation of the straight line which passes through the point (1, – 3, 2) and is parallel to the straight line $\frac{-x-1}{3} = \frac{y+4}{1} = \frac{2z-4}{2}$. (*JAC*, 2015)

Solution :

$$\text{Straight line } \frac{-x-1}{3} = \frac{y+4}{1} = \frac{2z-4}{2}$$

$$\Rightarrow \quad \frac{x+1}{-3} = \frac{y+4}{1} = \frac{z-2}{1}$$

Hence, direction ratio will be – 3, 1, 1.

Direction cosines of line will be

$$l = \frac{-3}{\sqrt{(-3)^2+1^2+1^2}},\ m = \frac{1}{\sqrt{(-3)^2+1^2+1^2}},$$

$$\text{and} \quad n = \frac{1}{\sqrt{(-3)^2+1^2+1^2}}$$

$$\Rightarrow \quad l = \frac{-3}{\sqrt{11}},\ m = \frac{1}{\sqrt{11}},\ n = \frac{1}{\sqrt{11}}$$

Hence, direction cosines are $\frac{-3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}$

Required line equation of given line :

$$\frac{x-1}{-3} = \frac{y+3}{1} = \frac{z-2}{1}$$

Q. 21. If $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$, then evaluate $|\vec{a}|$. (*USEB*, 2015)

Solution :

$$\because \quad \vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$$

$$\therefore \quad |\vec{a}| = |2\hat{i} + \hat{j} - 2\hat{k}|$$

$$= \sqrt{(2)^2+(1)^2+(-2)^2}$$

$$= \sqrt{4+1+4}$$

$$= \sqrt{9} = 3$$

Q. 22. Find the area of parallelogram whose adjacent sides are given by the vectors $\bar{a} = \hat{i} + \hat{j} + \hat{k}$ and $\bar{b} = 3\hat{i} + 2\hat{j} + \hat{k}$. (*USEB*, 2015)

Solution :

$\therefore$ Vector area of parallelogram $= \vec{a} \times \vec{b}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 3 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= \hat{i}(1-6) - \hat{j}(1-9) + \hat{k}(2-3)$$

$$= -5\hat{i} + 8\hat{j} - \hat{k}$$

Area of parallelogram

$$= |\vec{a} \times \vec{b}| = |-5\hat{i} + 8\hat{j} - \hat{k}|$$

$$= \sqrt{(-5)^2+8^2+(-1)^2} = \sqrt{90} = 3\sqrt{10} \text{ square units}$$

Q. 23. Find the equations of the straight line perpendicular to the two lines $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$; $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ and passing through their point of intersection. (*BSEB*, 2015)

Solution :

$$\text{Lines} \quad \frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1} = r_1 \quad \text{...(1)}$$

$$\text{and} \quad \frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2} = r_2 \quad \text{...(2)}$$

On line (1) point, A $(-3r_1 - 1, 2r_1 + 3, r_1 - 2)$ and on line (2) point B $(r_2, -3r_2 + 7, 2r_2 - 7)$.

from (1), Direction ratio of line, $a_1 = -3, a_2 = 2, a_3 = 1$

from (2), Direction ratio of line, $b_1 = 1, b_2 = -3, b_3 = 2$

Direction ratio of line passing through point A and B

$$C_1 = r_2 + 3r_1 + 1$$

$$C_2 = -3r_2 + 7 - (2r_1 + 3) = -3r_2 - 2r_1 + 4$$

$$C_3 = 2r_2 - 7 - (r_1 - 2) = 2r_2 - r_1 - 5$$

Line AB will be perpendicular to first line, if $a_1c_1 + a_2c_2 + a_3c_3 = 0$

$$\Rightarrow -3(r_2 + 3r_1 + 1) + 2(-3r_2 - 2r_1 + 4) + 1(2r_2 - r_1 - 5) = 0$$

$$\Rightarrow \quad -7r_2 - 14r_1 = 0$$

$$\Rightarrow \quad 7r_2 + 14r_1 = 0$$

$$\Rightarrow \quad r_2 + 2r_1 = 0 \quad \text{...(3)}$$

Line A and B will be perpendicular to second line if $b_1c_1 + b_2c_2 + b_3c_3 = 0$

$$1(r_2 + 3r_1 + 1) - 3(-3r_2 - 2r_1 + 4) + 2(2r_2 - r_1 - 5) = 0$$

$$14r_2 - 7r_1 - 21 = 0$$

$$\Rightarrow \quad 2r_2 - r_1 = 0 \quad \text{...(4)}$$

Solving equations (3) and (4)

$$r_1 = \frac{-3}{5} \text{ and } r_2 = \frac{3}{5}$$

$\therefore$ Point $\quad A = \left[-3\times\frac{-3}{5}-1, 2\times\frac{-3}{5}+3, \frac{-3}{5}-2\right]$

$$= \left(\frac{4}{5}, \frac{9}{5}, \frac{-13}{5}\right)$$

and point $\quad B = \left[\frac{3}{5}, -3\times\frac{3}{5}+7, 2\times\frac{3}{5}-7\right]$

$$= \left(\frac{3}{5}, \frac{6}{5}, \frac{-29}{5}\right)$$

Hence, equation of line AB

$$\frac{x-\frac{4}{5}}{\frac{3}{5}-\frac{4}{5}} = \frac{y-\frac{9}{5}}{\frac{6}{5}-\frac{9}{5}} = \frac{z+\frac{13}{5}}{\frac{-29}{5}+\frac{13}{5}}$$

$$\Rightarrow \quad \frac{5x-4}{-1} = \frac{5y-9}{-3} = \frac{5z+13}{-16}$$

Long Answer Type Questions

Q. 1. Find the equation of the plane perpendicular to the line joining the points (2, – 1, 2) and (3, 2, – 1) and passing through the point (4, – 3, 1).

(*JAC*, 2013)

Solution :

Any plane passing through the point (4, – 3, 1) is given by

$A(x-4) + B(y+3) + C(z-1) = 0$ ...(1)

where A, B, C are the direction ratio of the normal to the plane

DR's of the line joining the points (2, – 1, 2) and (3, 2, – 1)

$$= 3-2, 2+1, -1-2$$
$$= 1, 3, -3$$

Since the plane (1) is perpendicular to this line, therefore, the normal to this plane is parallel to this line.

So, $\frac{A}{1} + \frac{B}{3} + \frac{C}{-3} = \lambda$ (say)

$\Rightarrow \quad A = \lambda, B = 3\lambda, C = 3\lambda$

Putting these values of A, B and C in equation (1), it reduces to

$\lambda(x-4) + 3\lambda(y+3) - 3\lambda(z-1) = 0$

$\Rightarrow \quad (x-4) + 3(y+3) - 3(z-1) = 0$

$\Rightarrow \quad x-4+3y+9-3z+3 = 0$

$\Rightarrow \quad x+3y-3z+8 = 0$

which is the required equation of the plane.

Q. 2. Find the equations of the straight line perpendicular to the two lines $\frac{x+1}{-3} = \frac{y-3}{-2} = \frac{z+2}{1}$; $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ and passing through their point of intersection.

(*BSEB*, 2014)

Solution :

The given lines are

$$\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1} = r \text{ (say)} \quad ...(1)$$

and $$\frac{x}{1} = \frac{y-7}{-3} = \frac{y+7}{2} = R \text{ (say)} \quad ...(2)$$

Any point on line (1) is $(-1-3r, 3+2r, -2+r)$

Any point on line (2) is $(R, 7-3R, -7+2R)$

At the point of intersection for some values of r and R, we have

$$-1-3r = R$$
$$3+2r = 7-3R$$
$$-2+r = -7+2R$$

$\Rightarrow \quad 3r + R = -1$ ...(3)

$2r + 3R = 4$ ...(4)

$r - 2R = -5$ ...(5)

Solving equations (3) and (4), we get,

$$r = -1, R = 2$$

These values of r and R satisfy equation (5).

Hence the lines (1) and (2) intersect. Their point of intersection is given by

$(-1+3, 3-2, -2-1)$ or $(2, 1, -3)$

Let the d.c.'s of the line be $l, m, n,$ since the line is perpendicular to (1) and (2) both, therefore

$l(-3) + m(2) + n(1) = 0$ ...(6)

$l(1) + m(-3) + n(2) = 0$ ...(7)

From (6) and (7),

$$\frac{l}{4+3} = \frac{m}{1+6} = \frac{n}{9-2}$$

$$\Rightarrow \quad \frac{l}{7} = \frac{m}{7} = \frac{n}{7}$$

$$\Rightarrow \quad \frac{l}{1} = \frac{m}{1} = \frac{n}{1}$$

$\therefore$ DR's of the line are 1, 1, 1

Hence, the equation of the required line are

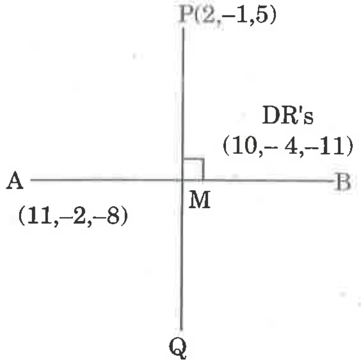
$$\frac{x-2}{1} = \frac{y-1}{1} = \frac{z+3}{1}$$

Q. 3. Find the image of the point (2, – 1, 5) in the line $\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11}$. Also, find the equations of the line joining the given point and its image. Find the length of that line segment also.

[*CBSE*, 2013 (*Comptt.*)]

Solution :

The given line is



$$\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11} = r \text{ (say)} \qquad \ldots(1)$$

It passes through A (11, – 2, – 8) and has dr' s 10, – 4, – 11

Let $p \rightarrow (2, -1, 5)$

Let M be the foot of the perpendicular drawn from point p to the line AB.

Any point on line (1) is $(11 + 10r, -2 - 4r, -8 - 11r)$

For some value of r, it will represent the point M.

DR' s of PM are $11 + 10r - 2, -2 - 4r + 1, -8 - 11r - 5$

i.e., $10r + 9, -4r - 1, -11r - 13$

$\because$ AB $\perp$ PM

$\therefore\ 10(10r + 9) - 4(-4r - 1) - 11(-11r - 13) = 0$

$\Rightarrow \quad 100r + 90 + 16r + 4 + 121r + 143 = 0$

$\Rightarrow \quad 237r + 237 = 0$

$\Rightarrow \quad r = -1$

$\therefore$ M = (11 – 10, – 2 + 4, – 8 + 11)

$\Rightarrow$ M $\rightarrow$ (1, 2, 3)

Let the image of the point P in the line (1) be $\theta(\alpha, \beta, \gamma)$, then

M is the mid-point of PQ

$$\therefore \quad \frac{\alpha+2}{2} = 1$$

$$\frac{\beta-1}{2} = 2$$

and
$$\frac{\gamma+3}{2} = 5$$

$\Rightarrow \quad \alpha = 0, \beta = 5, \gamma = 7$

Hence, the image point is (0, 5, 7)

Equations of the line joining the given point (2, – 1, 5) and its image (0, 5, 7) ane

$$\frac{x-2}{0-2} = \frac{y+1}{5-(-1)} = \frac{z-5}{7-5}$$

$$\Rightarrow \quad \frac{x-2}{-2} = \frac{y+1}{6} = \frac{z-5}{2}$$

$$\Rightarrow \quad \frac{x-2}{-1} = \frac{y+1}{3} = \frac{z-5}{1}$$

Also, length of the line segment :

$$= \sqrt{(2-0)^2 + (-1-5)^2 + (5-7)^2}$$

$$= \sqrt{4 + 36 + 4} = \sqrt{44}$$

$$= 2\sqrt{11} \text{ units}$$

Q. 4. Show that the lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$ intersect. Also, find their point of intersection.

[*CBSE,* 2013, 14 (*Comptt.*)]

Solution :

The given lines are $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) \ \ldots(1)$

and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k}) \qquad (2)$

If the lines (1) and (2) intersect, then at the point of intersection for some values of λ and μ, we have

$$(\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$$

$$\Rightarrow (1 + 3\lambda)\hat{i} + (1 - \lambda)\hat{j} - \hat{k} = (4 + 2\mu)\hat{i} + (3\mu - 1)\hat{k}$$

Equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$ on both sides, we get

$1 + 3\lambda = 4 + 2\mu$

$1 - \lambda = 0$

and $\quad -1 = 3\mu - 1$

$\Rightarrow \quad 3\lambda - 2\mu = 3 \qquad \ldots(3)$

$\lambda = 1 \qquad \ldots(4)$

$3\mu = 0 \qquad \ldots(5)$

Solving equations (4) and (5), we get

$\lambda = 1, \mu = 0$

These values of λ and μ clearly satisfy equation (1).

Hence the lines (1) and (2) intersect.

Their point of intersection is given by

$(\hat{i} + \hat{j} - \hat{k}) + 1(3\hat{i} - \hat{j}) = 4\hat{i} - \hat{k}$

Q. 5. Find the distance of the point (2, 12, 5) from the point of intersection of the line $\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0$.

(*AI CBSE,* 2014)

Solution :

The given line is $\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) \qquad \ldots(1)$

The equation of the plane is

$$\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0 \qquad \ldots(2)$$

At the point of intersection of line (1) and plane (2), for some value of λ, the value of $\vec{r}$ from (1) and (2) will be the same.

$\therefore\ \{(2\hat{i} - 4\hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})\}$
$\hat{i} - 2\hat{j} + \hat{k} = 0$

$\Rightarrow \quad (2 + 8 + 2) + \lambda(3 - 8 + 2) = 0$

$\Rightarrow \quad 12 - 3\lambda = 0$

$\Rightarrow \quad \lambda = 4$

Hence the points of intersection is :

$2\hat{i} - 4\hat{j} + 2\hat{k} + 4(3\hat{i} + 4\hat{j} + 2\hat{k})$

$= 14\hat{i} + 12\hat{j} + 10\hat{k}$

It represents the point (14, 12, 10)

Hence, the required distance

$$= \sqrt{(14-2)^2 + (12-12)^2 + (10-5)^2}$$

$$= \sqrt{144 + 0 + 25} = \sqrt{169}$$

= 13 units

Q. 6. Find the distance of the point (– 1, – 5, – 10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$. [*CBSE*, 2014; *AI CBSE*, 2014 (*Comptt.*)]

Solution :

The given line is

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) \quad ...(1)$$

The equation of the plane is

$$\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5 \quad ...(2)$$

At the point of intersection of line (1) and plane (2), for some value of λ, the values of $\vec{r}$ from (1) and (2) will be the same

$$\Rightarrow \{2i - j + 2k + \lambda(3i + 4j + 2k)\}(i - j + k) = 5$$

$$\Rightarrow (2 + 1 + 2) + \lambda(3 - 4 + 2) = 5$$

$$\Rightarrow 5 + \lambda = 5$$

$$\Rightarrow \lambda = 0$$

Hence, the point of intesection of line (1) and plane (2) is

$$2i - j + 2k$$

It represents the point (2, – 1, 2)

$\therefore$ Required distance

$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$$

$$= \sqrt{9 + 16 + 144}$$

$$= \sqrt{169}$$

$$= 13 \text{ units}$$

Q. 7. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$. Also find the distance of the plane obtained above from the orgin. (*AI CBSE*, 2014)

Solution :

Any plane passing through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ is given by $x + y + z - 1 + \lambda(2x + 3y + 4z - 5) = 0$ where λ is a parameter.

$\Rightarrow (1 + 2\lambda)r + (1 + 3\lambda)y + (1 + 4\lambda)z = 1 + 5\lambda \quad ...(1)$

If plane (1) is perpendicular the plane

$x - y + z = 0 \quad ...(2)$ then

$(1 + 2\lambda)(1) + (1 + 3\lambda)(-1) + (1 + 4\lambda)(1) = 0$

$\Rightarrow 1 + 2\lambda - 1 - 3\lambda + 1 + 4\lambda = 0$

$$\Rightarrow 3\lambda + 1 = 0 \Rightarrow \lambda = -\frac{1}{3}$$

puting this value of λ in (1), we get

$$(x + y + z - 1) - \frac{1}{3}(2x + 3y + 4z - 5) = 0$$

$$\Rightarrow 3(x + y + z - 1) - (2x + 3y + 4z - 5) = 0$$

$$\Rightarrow x - z + 2 = 0$$

which is the equation of the required plane.

Distance of this plane from origin

$$= \frac{0 - 0 + 2}{\sqrt{(1)^2 + (-1)^2}}$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2} \text{ units}$$

Q. 8. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r}.(3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from origin is unity.

[*AI CBSE*, 2013; *CBSE*, 2013 (*Comptt.*)]

Solution :

Any plane passing through the line of intersection of the planes $\vec{r}.(\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r}.(3\hat{i} - \hat{j} - 4\hat{k}) = 0$ is given by :

$\vec{r}.(\hat{i} + 3\hat{j}) - 6 + \lambda[\vec{r}.(3\hat{i} - \hat{j} - 4\hat{k})] = 0$, where λ is a parameter.

$$\Rightarrow \vec{r}.[(3\lambda + 1)\hat{i} + (3 - \lambda)\hat{j} - 4\lambda\hat{k}] - 6 = 0 \quad ...(1)$$

Its distance from (0) is 1

$$\therefore \left|\frac{[(3\lambda + 1)0 + (3 - \lambda)0 - 4\lambda(0) - 6]}{\sqrt{(3\lambda + 1)^2 + (3 - \lambda)^2 - (4\lambda)^2]}}\right| = 1$$

$$\Rightarrow 6 = \sqrt{(3\lambda + 1)^2 + (3 - \lambda)^2 + (-4\lambda)^2]}$$

Squaring, we get

$$36 = a\lambda^2 + 6\lambda + 1 + 9 + \lambda^2 - 6\lambda + 16\lambda^2$$

$$\Rightarrow 36 = 26\lambda^2 + 10$$

$$\Rightarrow 26\lambda^2 = 26$$

$$\Rightarrow \lambda^2 = 1$$

$$\Rightarrow \lambda = \pm 1$$

when $\lambda = 1$, (1) gives

$$\vec{r}.(4\hat{i} + 2\hat{j} - 4\hat{k}) - 6 = 0$$

$$\Rightarrow \vec{r}.(2\hat{i} + \hat{j} - 3\hat{k}) - 3 = 0 \quad ...(2)$$

when $\lambda = -1$, (1) gives

$$\vec{r}.(-2\hat{i} + 4\hat{j} + 4\hat{k}) - 6 = 0$$

$$\Rightarrow \vec{r}.(-\hat{i} + 2\hat{j} + 2\hat{k}) - 3 = 0 \quad ...(3)$$

(2) and (3) given the required planes.

Q. 9. Show that the lines : $\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$; $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$ are intersecting. Hence find their point of intersection. (*AI CBSE*, 2013)

Solution :

The given lines are

$$\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k}) \quad ...(1)$$

and

$$\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k}) \quad ...(2)$$

If the lines (1) and (2) and intersecting, then at their point of intersection, for some values of λ and μ, the values of $\vec{r}$ from (1) and (2) will be the same. Hence, at the point of intersection, we have

$3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k}) = 5\hat{i}-2\hat{j}+\mu(3\hat{i}+2\hat{j}+6\hat{k})$

$\Rightarrow (3+\lambda)\hat{i}+(2+2\lambda)\hat{j}+(4+2\lambda)\hat{k} = (5+3\mu)\hat{i}+(2\mu-2)\hat{j}+6\mu\hat{k}$

equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ on both sides, we get

$$3+\lambda = 5+3\mu \quad \ldots(3)$$
$$2+2\lambda = 2\mu-2 \quad \ldots(4)$$
$$-4+2\lambda = 6\mu \quad \ldots(5)$$
$$\Rightarrow \lambda-3\mu = 2 \quad \ldots(6)$$
$$\lambda-\mu = -2 \quad \ldots(7)$$
$$\lambda-3\mu = +2 \quad \ldots(8)$$

Solving (7) and (8), we get

$$\lambda = -4, \mu = -2$$

These values of λ and μ satisfy (6).

Hence the lines (1) and (2) intersect.

Their point of intersection is given by :

$3\hat{i}+2\hat{j}-4\hat{k}-4(\hat{i}+2\hat{j}+2\hat{k}) = -\hat{i}-6\hat{j}-12\hat{k}$,

i.e., the point (– 1, – 6, – 12).

Q. 10. Find the co-ordinates of the point whose the line through (3, – 4, – 5), (2, – 3, 1) crosses the plane, passing through the points (2, 2, 1), (3, 0, 1) and (4, – 1, 0). (*CBSE*, 2013)

Solution :

Any plane passing through the point (2, 2, 1) is given by

$$A(x-2)+B(y-2)+C(z-1) = 0 \quad \ldots(1)$$

where A, B, C are the direction ratios of the normal to the plane.

If plane (1) passes through the points (3, 0, 1) and (4, – 1, 0), then

$$A(1)+B(-2)+C(0) = 0 \quad \ldots(2)$$

and $$A(2)+B(-3)+C(-1) = 0 \quad \ldots(3)$$

Eliminating A, B, C from (1), (2) and (3) determinantically, we get

$$\begin{vmatrix} x-2 & y-2 & z-1 \\ 1 & -2 & 0 \\ 2 & -3 & -1 \end{vmatrix} = 0$$

$$\Rightarrow 2(x-2)+(y-2)+(z-1) = 0$$
$$\Rightarrow 2x+y+z = 7 \quad \ldots(4)$$

Equations of the line joining the points (3, – 4, – 5) and (2, – 3, 1) are

$$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = r \text{ (say)}$$

Any point on this line is $(3-r, -4+r, -5+6r)$

If it lies on (4), then

$$2(3-r)+(-4+r)+(-5+6r) = 7$$
$$\Rightarrow 6-2r-4+r-5+6r = 7$$
$$\Rightarrow 5r = 10$$
$$\Rightarrow r = 2$$

Hence the point of intersection is (3 – 2, – 4 + 2, – 5 + 12) or (1, – 2, 7)

Q. 11. Find the vector equation of the plane determined by the points A (3, – 1, 2), B (5, 2, 4) and C (– 1, – 1, 6). Also, find the distance of point P (6, 5, 9) from this plane. (*CBSE*, 2013)

Solution :

Any plane passing through A (3 – 1, 2) is given by

$$a(x-3)+b(y+1)+c(z-2) = 0 \quad \ldots(1)$$

where a, b, c are the d.r.'s of the normal to the plane.

If plane (1) passes through B and C, then

$$a(2)+b(3)+c(2) = 0 \quad \ldots(2)$$
$$a(-4)+b(0)+c(4) = 0 \quad \ldots(3)$$

Eliminating a, b, c determinantically from (1), (2) and (3), we get

$$\begin{vmatrix} x-3 & y+1 & z-2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 0$$

$$\Rightarrow 12(x-3)-16(y+1)+12(z-2) = 0$$
$$\Rightarrow 3(x-3)-4(y+1)+3(z-2) = 0$$
$$\Rightarrow 3x-4y+3z-19 = 0 \quad \ldots(4)$$

Its vector equation is $\vec{r}.(3\hat{i}-4\hat{j}+3\hat{k})-19 = 0 \ldots(5)$

$$\text{Distance of (4) from } p = \frac{3(6)-4(5)+3(9)-19}{\sqrt{(3)^2+(-4)^2+(3)^2}}$$
$$= \frac{18-20+27-19}{\sqrt{9+16+9}}$$
$$= \frac{6}{\sqrt{34}} \text{ units}$$

Q. 12. Find the vector equation of the plane passing through three points with position vectors $\hat{i}+\hat{j}-2\hat{k}$, $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$. Also find the co-ordinates of the point of intersection of this plane and the line $\vec{r} = 3\hat{i}-\hat{j}-\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k})$. (*CBSE*, 2013)

Solution :

Here, $\vec{a} = \hat{i}+\hat{j}-2\hat{k}$

$\vec{b} = 2\hat{i}-\hat{j}+\hat{k}$

and $\vec{c} = \hat{i}+2\hat{j}+\hat{k}$

$$\therefore \vec{b}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$
$$= -3\hat{i}-\hat{j}+5\hat{k}$$

Equation of the plane is $(\vec{r}-\vec{a}).(\vec{b}\times\vec{c}) = 0$

$$\Rightarrow \{\vec{r}-(\hat{i}+\hat{j}-2\hat{k})\}.(-3\hat{i}-\hat{j}+5\hat{k}) = 0$$

$$\Rightarrow \vec{r}.(-3\hat{i}-\hat{j}+5\hat{k}) = (\hat{i}+\hat{j}-2\hat{k}).(-3\hat{i}-\hat{j}+5\hat{k})$$
$$= -3-1-10$$
$$= -14$$

$\Rightarrow \quad \vec{r}.(3\hat{i} + \hat{j} - 5\hat{k}) = 14 \quad ...(1)$

The line is $\vec{r} = 3\hat{i} - \hat{j} - \hat{k} + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ ...(2)

The point of intersection of (1) and (2), we have

$[3\hat{i} - \hat{j} - \hat{k} + \lambda(2\hat{i} - 2\hat{j} + \hat{k})].(3\hat{i} + \hat{j} - 5\hat{k}) = 14$

$\Rightarrow \quad (9 - 1 + 5) + \lambda(6 - 2 - 5) = 14$

$\Rightarrow \quad 13 - \lambda = 14$

$\Rightarrow \quad \lambda = -1$

Hence, the point of intersection of (1) and (2) is

$3\hat{i} - \hat{j} - \hat{k} + (-1)(2\hat{i} - 2\hat{j} + \hat{k})$

$= \hat{i} + \hat{j} - 2\hat{k}$

Q. 13. Find the equation of the plane which contains the line of intersection of the planes $\vec{r}.(\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0$ and $\vec{r}.(2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$ and which is perpendicular to the plane $\vec{r}.(5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$. (*RSEB*, 2013; *CBSE*, 2013)

Solution :

Any plane through the line of intersection of the planes

$\vec{r}.(\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0 \quad ...(1)$

and $\vec{r}.(2\hat{i} + \hat{j} - \hat{k}) + 5 = 0 \quad ...(2)$

is given by

$\vec{r}.(\hat{i} + 2\hat{j} + 3\hat{k}) - 4 + \lambda[\vec{r}.(2\hat{i} + \hat{j} - \hat{k}) + 5] = 0$

$\Rightarrow \vec{r}[(1 + 2\lambda)\hat{i} + (2 + \lambda)\hat{j} + (3 - \lambda)\hat{k}] + 5\lambda - 4 = 0 \quad ...(3)$

(3) is perpendicular to the plane

$\vec{r}.(5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0 \quad ...(4)$

$\therefore \quad (1 + 2\lambda)5 + (2 + \lambda)3 + (3 - \lambda)(-6) = 0$

$\Rightarrow \quad 5 + 10\lambda + 6 + 3\lambda - 18 + 6\lambda = 0$

$\Rightarrow \quad 19\lambda - 7 = 0$

$\Rightarrow \quad \lambda = \frac{7}{19}$

Putting this value of λ in (3), we get

$\vec{r}.\left\{\left(1 + \frac{14}{19}\right)\hat{i} + \frac{45}{19}\hat{j} + \frac{50}{19}\hat{k}\right\} + \frac{35}{19} - 4 = 0$

$\Rightarrow \quad \vec{r}.(33\hat{i} + 45\hat{j} + 50\hat{k}) - 41 = 0$

which is the required equation of the plane.

Q. 14. Find the co-ordinates of the point where the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$ intersects the plane $x - y + z - 5 = 0$. Also, find the angle between the line and the plane. (*CBSE*, 2013)

Solution :

The given line is

$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = r \text{ (say)}$

Any point on this line is $(2 + 3r, -1 + 4r, 2 + 2r)$

It lies on the plane

$x - y + z - 5 = 0$, then

$2 + 3r - 1 - 4r + 2 + 2r - 5 = 0$

$\Rightarrow \quad -5r = 0$

$\Rightarrow \quad r = 0$

$\therefore$ the point of intersection is $(2, -1, 2)$

Let the required angle be θ, then

$$\sin\theta = \frac{(3)(1) + 4(-1) + (2)(1)}{\sqrt{3^2 + 4^2 + 2^2}\sqrt{1^2 + (-1)^2 + 1^2}}$$

$$= \frac{1}{\sqrt{87}}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{\sqrt{87}}\right)$$

Q. 15. Find the vector equation of the plane passing through the intersection of the planes $\vec{r}(\hat{i} + \hat{j} + \hat{k}) = 6$ and $\vec{r}(2\hat{i} + 3\hat{j} + 4\hat{k}) = -5$ and the point (1, 1, 1). (*USEB*, 2013)

Solution :

Any plane passing through the intersection of the planes $\vec{r}.(\hat{i} + \hat{j} + \hat{k}) = 6$ and $\vec{r}.(2\hat{i} + 3\hat{j} + 4\hat{k}) = -5$ is given by

$\vec{r}.(\hat{i} + \hat{j} + \hat{k}) - 6 + \lambda[\vec{r}.(2\hat{i} + 3\hat{j} + 4\hat{k}) + 5] = 0 \quad ...(1)$

where λ is a parameter.

$\Rightarrow \vec{r}.[(1 + 2\lambda)\hat{i} + (1 + 3\lambda)\hat{j} + (1 + 4\lambda)\hat{k}] + 5\lambda - 6 = 0 \quad ...(2)$

If (2) passes through (1, 1, 1), *i.e.*, $\hat{i} + \hat{j} + \hat{k}$, then

$(\hat{i} + \hat{j} + \hat{k}).[(1 + 2\lambda)\hat{i} + (1 + 3\lambda)\hat{j} + (1 + 4\lambda)\hat{k}] + 5\lambda - 6 = 0$

$\Rightarrow 1 + 2\lambda + 1 + 3\lambda + 1 + 4\lambda + 5\lambda - 6 = 0$

$\Rightarrow \quad 14\lambda = 3$

$\Rightarrow \quad \lambda = \frac{3}{14}$.

Putting the value of λ in (2), we get

$\vec{r}.\left[\left(1 + \frac{3}{7}\right)\hat{i} + \left(1 + \frac{9}{14}\right)\hat{j} + \left(1 + \frac{6}{7}\right)\hat{k}\right] + \frac{15}{14} - 6 = 0$

$\Rightarrow \vec{r}.\left[\frac{10}{7}\hat{i} + \frac{23}{14}\hat{j} + \frac{13}{7}\hat{k}\right] - \frac{69}{14} = 0$

$\Rightarrow \vec{r}.(20\hat{i} + 23\hat{j} + 26\hat{k}) - 69 = 0$

which is the required vector equation of the plane.

Q. 16. Find vector equation of a plane passing through the intersection of the planes $\vec{r}.(2\hat{i} + 2\hat{j} + 3\hat{k}) = 7$ and $\vec{r}.(2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and the point (2, 1, 3). (*Raj. Board*, 2014)

Solution :

Any plane passing through the line of intersection of the planes

$$\vec{r}.(2\hat{i}+2\hat{j}+3\hat{k}) = 7 \quad ...(1)$$

and $\vec{r}.(2\hat{i}+5\hat{j}+3\hat{k}) = 9$ is given by ...(2)

$$\vec{r}.(2\hat{i}+2\hat{j}+3\hat{k}) - 7 + \lambda[\vec{r}.(2\hat{i}+5\hat{j}+3\hat{k}) - 9] = 0 \quad ...(3)$$

where λ is a parameter.

$$\Rightarrow \vec{r}.[(2+2\lambda)\hat{i} + (2+5\lambda)\hat{j} + (3+3\lambda)\hat{k}] = 7+9\lambda \quad ...(4)$$

If it passes through the point (2, 1 ,3) *i.e.,* $2\hat{i} + \hat{j} + 3\hat{k}$, then

$$\Rightarrow (2\hat{i} + \hat{j} + 3\hat{k})[(2+2\lambda)\hat{i} + (2+5\lambda)\hat{j} + (3+3\lambda)\hat{k}] = 7+9\lambda$$

$$\Rightarrow 2(2+2\lambda) + 1(2+5\lambda) + 3(3+3\lambda) = 7+9\lambda$$

$$\Rightarrow 2+4\lambda+2+5\lambda+9+9\lambda = 7+9\lambda$$

$$\Rightarrow 9\lambda + 6 = 0$$

$$\Rightarrow \lambda = -\frac{2}{3}$$

Putting the value of λ in (4), we get

$$\vec{r}.\left[\left(2-\frac{4}{3}\right)\hat{i}+\left(2-\frac{10}{3}\right)\hat{j}+(3-2)\hat{k}\right] = 7-6$$

$$\Rightarrow \vec{r}.\left(\frac{2}{3}i-\frac{4}{3}j+\hat{k}\right) = 1$$

$$\Rightarrow \vec{r}.(2\hat{i} - 4\hat{j} + 3\hat{k}) = 3$$

which is the required equation of the plane.

Q. 17. Find the shortest distance between the lines $\frac{x+3}{-4} = \frac{y-6}{3} = \frac{z}{2}$ and $\frac{x+2}{-4} = \frac{y}{1} = \frac{z-7}{1}$.

(JAC, 2013)

Solution :

$$\text{S.D.} = \frac{\begin{vmatrix} (-2)-(-3) & 0-6 & 7-0 \\ -4 & 3 & 2 \\ -4 & 1 & 1 \end{vmatrix}}{\sqrt{(3-2)^2+(-8+4)^2+(-4+12)^2}}$$

$$= \frac{\begin{vmatrix} 1 & -6 & 7 \\ -4 & 3 & 2 \\ -4 & 1 & 1 \end{vmatrix}}{\sqrt{1+16+64}}$$

$$= \frac{1(3-2)-6(-8+4)+7(-4+12)}{9}$$

$$= \frac{1+24+56}{9} = \frac{81}{9}$$

$= 9$ units

Q. 18. Consider the equations of the straight lines given by :

$$\mathbf{L_1 : \vec{r} = (\hat{i}+2\hat{j}+\hat{k}) + \lambda(\hat{i}-\hat{j}+\hat{k})}$$

$$\mathbf{L_2 : \vec{r} = (2\hat{i}-\hat{j}-\hat{k}) + \mu(2\hat{i}+\hat{j}+2\hat{k})}$$

Find the shortest distance between L_1 and L_2.

(JAC, 2014)

Solution :

$$\vec{a}_1 = \hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{a}_2 = 2\hat{i} - \hat{j} - \hat{k}$$

$$\vec{b}_1 = \hat{i} - \hat{j} + \hat{k}$$

and $$\vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = \hat{i} - 3\hat{j} - 2\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$= -3\hat{i} + 3\hat{k}$$

$$\text{S.D.} = \left|\frac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

$$= \left|\frac{(-3\hat{i}+3\hat{k}).(\hat{i}-3\hat{j}-2\hat{k})}{\sqrt{(-3)^2+(3)^2}}\right|$$

$$= \left|\frac{-3+0-6}{3\sqrt{2}}\right| = \left|-\frac{3}{\sqrt{2}}\right|$$

$$= \frac{3}{\sqrt{2}} \text{ units}$$

Q. 19. Find the shortest distance between the lines

$$\mathbf{\vec{r} = (4\hat{i}-\hat{j}) + \lambda(\hat{i}+2\hat{j}-3\hat{k}) \text{ and } \vec{r} = (\hat{i}-\hat{j}+2\hat{k}) + \mu(2\hat{i}+4\hat{j}-5\hat{k})}.$$

(BSEB, 2014)

Solution :

$$\vec{a}_1 = 4\hat{i} - \hat{j}$$

$$\vec{a}_2 = \hat{i} - \hat{j} + 2\hat{k}$$

$$\vec{b}_1 = \hat{i} + 2\hat{j} - 3\hat{k}$$

and $$\vec{b}_2 = 2\hat{i} + 4\hat{j} - 5\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = -3\hat{i} + 2\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$

$$= 2\hat{i} + 2\hat{j}$$

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(2)^2+(2)^2} = 2\sqrt{2}$$

$$\text{S.D.} = \left|\frac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

$$= \left|\frac{(2\hat{i} + 2\hat{j}).(-3\hat{i} + 2\hat{k})}{2\sqrt{2}}\right|$$

$$= \left|-\frac{6}{2\sqrt{2}}\right| = \left|-\frac{3}{\sqrt{2}}\right|$$

$$= \frac{3}{\sqrt{2}} \text{ units}$$

Q. 20. Find the shortest distance between the two lines whose vector equations are $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$ and $\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$.

[*CBSE*, 2014 (*Comptt.*) ; *USEB*, 14]

Solution :

$$\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$$

$$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$$

and $$\vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(-9)^2 + (3)^2 + (9)^2}$$

$$= \sqrt{81 + 9 + 81} = \sqrt{171}$$

$$= 3\sqrt{19}$$

$$\text{S.D.} = \left|\frac{(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

$$= \left|\frac{(-9\hat{i} + 3\hat{j} + 9\hat{k}).(3\hat{i} + 3\hat{j} + 3\hat{k})}{3\sqrt{19}}\right|$$

$$= \left|\frac{-27 + 9 + 27}{3\sqrt{19}}\right| = \left|\frac{3}{\sqrt{19}}\right|$$

$$= \frac{3}{\sqrt{19}} \text{ square units.}$$

NCERT QUESTIONS

Q. 1. Find the equation of plane passing through the point (–1, 3, 2) and ⊥ to each of the planes and $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

(*CBSE, Delhi and USEB*, 2009)

Solution :

Equation of plane passing through the point (–1, 3, 2)

$a(x + 1) + b(y - 3) + c(z - 2) = 0$ ...(1)

Plane (1) is perpendicular to planes $x + 2y + 3z = 5$, $3x + 3y + z = 0$.

Thus $a + 2b + 3c = 0$ ...(2)

and $3a + 3b + c = 0$ ...(3)

Solving eqs. (2) and (3),

$$\frac{a}{2-9} = \frac{b}{9-1} = \frac{c}{3-6} = \lambda \quad \text{(Let)}$$

$$\Rightarrow \quad \frac{a}{-7} = \frac{b}{8} = \frac{c}{-3} = \lambda$$

$$\Rightarrow \quad a = -7\lambda, b = 8\lambda, c = -3\lambda$$

Required equation of plane

$$-7\lambda(x + 1) + 8\lambda(y - 3) - 3\lambda(z - 2) = 0$$

$$\Rightarrow \quad -7(x + 1) + 8(y - 3) - 3(z - 2) = 0$$

$$a = -7x + 8y - 3z = 25.$$

Q. 2. The cartesian equation of a line are $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$, write its vector form.

Solution :

Equations of given lines are

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{x-6}{2}$$

show that the given line passes through the point A (5, – 4, 6) and parallel to the vector $\vec{m} = 3\hat{i} + 7\hat{j} + 2\hat{k}$:

P.V. of A, $\vec{r}_1 = 5\hat{i} - 4\hat{j} + 6\hat{k}$

∴ Vector equation of given line

$$\vec{r} = \vec{r}_1 + \lambda\vec{m}$$

$$\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k}).$$

Q. 3. Find the S.D. between the lines whose vector equations are :

$$\vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - 2t)\hat{k}$$

and $$\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}$$

Solution :

Given equation written as :

$$\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k})$$

and $$\vec{r} = (\hat{i} - 2\hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k})$$

Comparing with $\vec{r} = \vec{r}_1 + t\vec{u}$.

and $\vec{r} = \vec{r}_2 + s\vec{u}$,

$$\vec{r}_1 = (\hat{i} - 2\hat{j} + 3\hat{k}), \vec{r}_2 = (\hat{i} - \hat{j} - \hat{k})$$

$$\vec{u} = (-\hat{i} + \hat{j} - 2\hat{k}) \text{ and } \vec{u} = (\hat{i} + 2\hat{j} - 2\hat{k})$$

$$\therefore \quad (\vec{r}_2 - \vec{r}_1) = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k})$$

$$= \hat{j} - 4\hat{k}$$